Lab 4. Families of discrete distributions

sgr. 3

Exercise 54. An archer hits a bull's-eye with the probability of 0.09 and the results of different attempts can be taken as independent of each other. If the archer shoots 9 arrows, calculate the probability that: a) exactly two arrows score bull's-eyes; b) at least two arrows score bull's-eyes. What is the expected number of the bull's-eyes scored?

X - nr. of bull's-eyes scored

$X \sim Bin(n, p)$ $\quad n = 9$

$p = 0.09$ - probability of a "success" (hitting a bull's-eye)

a) $P(X=2) = \binom{9}{2} p^2 (1-p)^{9-2} = 36 \cdot 0.09^2 \cdot 0.91^7 = 0.15$

b) $P(X \geq 2) = 1 - P(X<2) = 1 - [P(X=0) + P(X=1)] = 1 - [\overset{1}{\binom{9}{0}} p^0 (1-p)^9 + \overset{9}{\binom{9}{1}} p^1 (1-p)^8] =$

$= 1 - [0.91^9 + 9 \cdot 0.09 \cdot 0.91^8] = 0.191$

$E[X] = np(1-p) = 9 \cdot 0.09 \cdot 0.91 = 0.737$

2. A representative from the National Football League's Marketing Division randomly selects people on a random street in Kansas City, Missouri until he finds a person who attended the last home football game. Let p, the probability that he succeeds in finding such a person, equal 0.20. And, let X denote the number of people he selects until he finds his first success. What is the probability mass function of X?

$X \sim Geom(p)$

$p = 0.20$ – probab. of finding a person that attended the last home football game

$$X : \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ & & & & (1-p)^{k-1} \cdot p & \end{pmatrix} \qquad P(X=k) = (1-p)^{k-1} \cdot p$$

PMF: $f : \mathbb{R} \to [0,1]$, $f(k) = (1-p)^{k-1} \cdot p$, $k \in \mathbb{N}^*$

$f(x) = 0$, $(\forall)\, x \notin \mathbb{N}^*$

$$P(X=4) = P(X=1) + P(X=2) + P(X=3) + P(X=4) = p + (1-p) \cdot p + (1-p)^2 \cdot p + (1-p)^3 \cdot p = 0.59$$

probab. of selecting 4 people before we find someone that attended the football game

Exercise 58. Two players A and B take part at a competition. Each turn of the competition may be won by player A with probability p or by player B with probability $(1-p)$. The competition ends either when A wins m turns or when B wins n turns. Find the probability that player A wins the competition at turn k.

$$\underbrace{\begin{array}{ccccc} A & B & B & \dots & A \\ 1 & 2 & 3 & & k-1 \end{array}}_{m-1\ A\text{s}} \begin{array}{c} A \\ k \end{array} \qquad k \geq m$$

fixed nr. of successes

X - nr. of turns played until A wins $\underline{m}$ turns (A wins the competition)

$X \sim NB(m, p)$

$$P(X=k) = \binom{k-1}{m-1} \cdot p^m \cdot (1-p)^{k-m}$$

Exercise 64. A baker blends 600 raisins and 400 chocolate chips into a mix and, from this, makes 500 cookies. (a) Find the probability that a randomly picked cookie will have no raisins. (b) Find the probability that a randomly picked cookie will have exactly two chocolate chips. (c) Find the probability that a randomly chosen cookie will have at least two bits (raisins or chips) in it.

X – nr. of raisins in a cookie

$X \sim Bin(n, p)$

$\downarrow$

$Po(\lambda)$, λ – average nr. of raisins in a cookie

$\lambda = \frac{600}{500} = \frac{6}{5}$ $\quad (\lambda = n \cdot p)$

a) $P(X=0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-\lambda} = e^{-\frac{6}{5}} = 0.301$

b) Y – nr. of chocolate chips in a cookie

$Y \sim Po(\lambda')$, $\lambda' = \frac{400}{500} = \frac{4}{5}$

$P(Y=2) = e^{-\lambda'} \cdot \frac{\lambda'^2}{2!} = 0.144$

c) Z – nr. of bits in a cookie

$Z \sim Po(\lambda'')$, $\lambda'' = \frac{600+400}{500} = 2$

$P(Z \geq 2) = 1 - P(Z < 2) =$

$= 1 - [P(Z=0) + P(Z=1)] =$

$= 1 - \left[e^{-2} \cdot 1 + e^{-2} \cdot \frac{2^1}{1!}\right] = 1 - 3e^{-2} =$

$= 0.594$

Ch.2 ex.5

A: the first card is the Ace of Spades

B: the second card is the 8 of Clubs

C: the third card is an ace

$$P(C \mid A \cap B) = \frac{P(C \cap A \cap B)}{P(A \cap B)} = \frac{P(C \cap A \cap B)}{P(A \cap B)} = \frac{\frac{3}{52 \cdot 51 \cdot 50}}{\frac{1}{52 \cdot 51}} = \frac{3}{50}$$

$$P(A \cap B) = \frac{1 \cdot 1}{52 \cdot 51} = \frac{1}{52 \cdot 51}$$

$$P(A \cap B \cap C) = \frac{1 \cdot 1 \cdot 3}{52 \cdot 51 \cdot 50} = \frac{3}{52 \cdot 51 \cdot 50}$$

$$P(A) = \frac{1}{52}$$

$$P(C \mid A \cap B) = \frac{3}{50}$$

(32) A: 4, 4, 4, 4, 0, 0
B: 3, 3, 3, 3, 3, 3
C: 6, 6, 2, 2, 2, 2

$P(A>B) = \frac{4}{6}$

D: 5, 5, 5, 1, 1, 1

a) $P(A>B) = P(A=4) = \frac{4}{6} \left(= \frac{2}{3}\right)$

$P(B>C) = P(C=2) = \frac{4}{6}$

$P(C>D) = P(C=6) + P(C=2, D=1) = \frac{2}{6} + P(C=2) \cdot P(D=1) =$

$= \frac{1}{3} + \frac{\cancel{4}^{2}}{\cancel{6}} \cdot \frac{3}{6} = \frac{1}{3} + \frac{2}{6} = \frac{4}{6}$

$P(D>A) = P(D=5) + P(A=0, D=1) = \frac{3}{6} + P(A=0) \cdot P(D=1) = \frac{1}{2} + \frac{\cancel{2}}{6} \cdot \frac{\cancel{3}}{6} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6}$

b) $A>B$? $B>C$

$B>C$? $C>D$

$A>B \equiv (A=4)$

$B>C \equiv (C=2)$ > independent events

not indep. $C>D \equiv (C=6)$ or $(C=2$ and $D=1)$

Exercise 52. On the average, only 1 person in 1000 has a particular rare blood type. (a) Find the probability that, in a city of 10,000 people, no one has this blood type. (b) How many people would have to be tested to give a probability greater than 1/2 of finding at least one person with this blood type?

sgr. 2

$x<1$
$\ln(x)<0$

a) X – nr. of people in the city that have this blood type

$X \sim Bin(n, p)$, $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$n = 10\,000$

$p = \frac{1}{1000}$ – probability that a person has the rare blood type (probab. of "success")

$p = 0.001$

$$P(X=0) = \binom{10\,000}{0} \cdot p^0 \cdot (1-p)^{10\,000} = 1 \cdot (1-p)^{10\,000} = 0.999^{10\,000} = 4.52 \cdot 10^{-5}$$

(b) $P(X \geq 1) > \frac{1}{2}$ $\iff 1 - P(X<1) > \frac{1}{2} \iff P(X<1) < \frac{1}{2} \iff P(X=0) < \frac{1}{2} \iff$

$n = ?$

$$\underbrace{\binom{n}{0}}_{1} \cdot \underbrace{p^0}_{1} \cdot (1-p)^n < \frac{1}{2} \iff 0.999^n < \frac{1}{2} \iff n \ln(0.999) < \ln\left(\frac{1}{2}\right) \iff$$

$$n > \frac{\ln(1/2)}{\ln(0.999)} = 692.8, \quad n \geq 693$$

2. You play a game of chance that you can either win or lose (there are no other possibilities) **until** you lose. Your probability of losing is $p = 0.57$. What is the probability that it takes five games until you lose?

X - nr. of games played until I lose

$X \sim \text{Geom}(p)$

"success" = losing the game

$p = 0.57$

W W W W L
1 2 3 4 5

$$P(X=5) = (1-p)^4 \cdot p = 0.43^4 \cdot 0.57 = 0.019$$

Exercise 59. Consider the same situation as in Ex. 5. (a) If the archer shoots a series of arrows, what is the probability that the first bull's-eye is scored with the fourth arrow? (b) What is the probability that the third bull's-eye is scored with the tenth arrow? (c) What is the expected number of the arrows shot before the first bull's-eye is scored? (d) What is the expected number of the arrows shot before the third bull's-eye is scored?

The archer hits a bull's-eye with a probability of 0.09.

a) X - nr. of arrow needed to score the first bull's-eye

$X \sim Geom(p)$

$p = 0.09$ (probab. of hitting a bull's-eyes = "success")

$P(X=4) = (1-p)^3 \cdot p = 0.91^3 \cdot 0.09 = 0.068$

b) Y - nr. of arrows needed to score 3 bull's-eyes

$Y \sim NB(k, p)$

$p = 0.09$, $k = 3$

$$\underbrace{\underset{1}{\overset{1}{B}}\ \underset{2}{O}\ \underset{3}{B} \ldots \underset{9}{O}}\ \underset{10}{B}$$

$P(Y=10) = \binom{9}{2} \cdot 0.09^3 \cdot 0.91^7 = 36 \cdot 0.09^3 \cdot 0.91^7 = 0.014$

c) $E[X] = \frac{1}{p} = \frac{1}{0.09} = \frac{100}{9} = 11.11$

$X \sim Geom(p)$

d) $E[Y] = \frac{k}{p} = \frac{3}{0.09} = 33.33$

Exercise 65 (London bombing). The statistics of flying bomb hits in an area in the south of London during the Second World War provide the following data. The area in question was divided into 24 x 24 = 576 small areas. The total number of hits was 537. There were 229 squares with 0 hits, 211 with 1 hit, 93 with 2 hits, 35 with 3 hits, 7 with 4 hits, and 1 with 5 or more. Assuming the hits were purely random, use the Poisson approximation to find the probability that a particular square would have exactly k hits. Compute the expected number of squares that would have 0, 1, 2, 3, 4, and 5 or more hits and compare this with the observed results.

X - nr. of hits for a particular square

$X \sim Bin(n, p)$

$n = 537$, $p = ?$ (probab. of a hit)

$X \approx Po(\lambda)$, $\lambda = n \cdot p$

λ - average of hits in a particular square

$\lambda = \frac{537}{576} = 0.932$

$P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$k=0 \Rightarrow P(X=0) = e^{-\lambda} = e^{-0.932} = 0.394$

$P(X=1) = e^{-\lambda} \cdot \frac{\lambda}{1!} = \lambda e^{-\lambda} = 0.367$

$P(X=2) = e^{-\lambda} \cdot \frac{\lambda^2}{2!} = 0.171$

$P(X=3) = e^{-\lambda} \cdot \frac{\lambda^3}{3!} = 0.053$

$P(X=4) = e^{-\lambda} \cdot \frac{\lambda^4}{4!} = 0.012$

$P(X \geq 5) = 1 - P(X < 5) =$

$= 1 - [P(X=0) + P(X=1) + \dots + P(X=4)]$

$= 0.003$

Nr. of squares with:

- 0 hits: $0.394 \cdot 576 = 226.94$
- 1 hits: 211.39
- 2 hits: 98.49
- 3 hits: 30.53
- 4 hits: 6.91
- ≥ 5 hits: 1.73

Ch. 1

(40) - n labelled balls in a jar

- k balls are selected (with replacement)

a) $P(\text{"strictly increasing sequence"}) = \dfrac{\frac{n(n-1)\dots(n-k+1)}{k!}}{n^k} = \dfrac{\binom{n}{k}}{n^k}$

$1<2<5< \dots$

- choose k distinct balls out of n balls $= \binom{n}{k}$

$\hookrightarrow$ choose k balls out of n balls without replacement

5, 2, 1

2, 1, 5

b) $P(\text{"increasing sequence"}) = \dfrac{\binom{n+k-1}{k}}{n^k}$

• | • • | ... | | •

"Bose-Einstein"

1, 1, 2, 3, 3, 4, 5, 5, 5

• 3, 4, 1, 2, 1, 4, 4, 5, 5, 3, 5

55) 10 - seniors

12 - juniors → choose a committee of 5 people.

15 - sophomores

a) $P(\text{"3 sophomores"}) = \dfrac{\binom{15}{3}\cdot\binom{22}{2}}{\binom{37}{5}}$

b) A: the committee has at least one senior

B: the — " — — " — one junior

C: — " — — " — one sophomore

$$P(A\cap B\cap C) = 1 - P(\overline{A\cap B\cap C}) = 1 - \underbrace{P(\bar A\cup\bar B\cup\bar C)} = 1 - 0.358 = 0.642$$

$$P(\bar A\cup\bar B\cup\bar C) = P(\bar A)+P(\bar B)+P(\bar C) - P(\bar A\cap\bar B) - P(\bar A\cap\bar C) - P(\bar B\cap\bar C) + P(\bar A\cap\bar B\cap\bar C) = 0.358$$

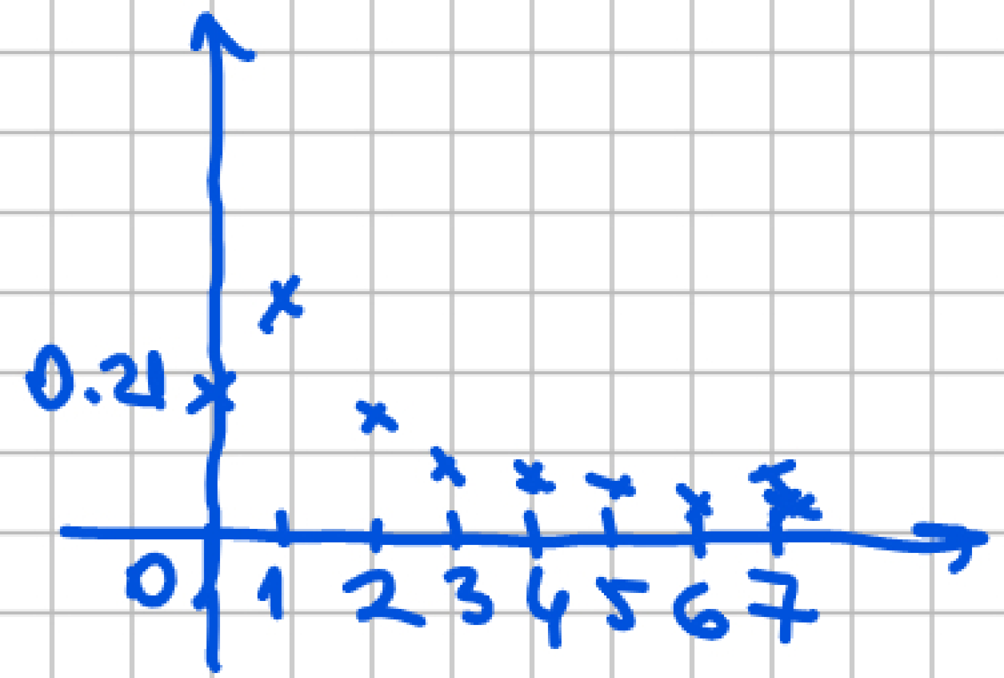
$P(\bar A) = \dfrac{\binom{27}{5}}{\binom{37}{5}}$, $P(\bar B) = \dfrac{\binom{25}{5}}{\binom{37}{5}}$, $P(\bar C) = \dfrac{\binom{22}{5}}{\binom{37}{5}}$, $P(\bar A\cap\bar B) = \dfrac{\binom{15}{5}}{\binom{37}{5}}$, $P(\bar A\cap\bar C) = \dfrac{\binom{12}{5}}{\binom{37}{5}}$ "0", $P(\bar B\cap\bar C) = \dfrac{\binom{10}{5}}{\binom{37}{5}}$

Exercise 51. The statistical data of a hospital shows that 20% of the patients suffering from a given illness die. If 7 patients have been diagnosed with this illness, what is the probability that:
a) all patients get well? b) at least 3 patients die? c) 4 patients die?

sgr. 5

X - nr. of patients that die

$X \sim Bin(n, p)$ $\qquad P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$



a) $P(X=0) = \underbrace{\binom{7}{0}}_{=1} \cdot p^0 \cdot (1-p)^{7-0} = (1-p)^7 = 0.80^7 = 0.21$

dbinom(0,7,0.2)

$n = 7$

$p = 20\% = 0.20$

b) $P(X \geq 3) = P(X=3) + P(X=4) + \dots + P(X=7)$

$= 1 - P(X<3) = 1 - [P(X=0) + P(X=1) + P(X=2)] =$

$= 1 - [0.21 + \underbrace{\binom{7}{1}}_{7} \cdot 0.2^1 \cdot 0.8^6 + \underbrace{\binom{7}{2}}_{=21} \cdot 0.2^2 \cdot 0.8^5] = 0.148$

c) $P(X=4) = \binom{7}{4} 0.2^4 \cdot 0.8^3 = 0.029$

dbinom(4,7,0.2)

$Bin(n,p)$ - toss a coin n times & count nr. of Heads

$Geom(p)$ - count the nr. of tosses needed to get 1st Heads

$NB(k,p)$ - nr. of tosses needed to get $\underline{k}$ Heads

$Po(\lambda)$ - nr. of telephone calls in an hour

2. You throw darts at a board until you hit the center area. Your probability of hitting the center area is $p = 0.17$. You want to find the probability that it takes eight throws until you hit the center. What values does X take on?

X - nr. throws until you hit the center

$$\underbrace{\underset{1}{F}\ \underset{2}{F}\ \underset{3}{F}\ \dots\ \underset{k-1}{F}}_{(1-p)^{k-1}}\ \cdot\ \underbrace{\underset{k}{C}}_{p}$$

$X \sim \text{Geom}(p)$, $P(X=k) = (1-p)^{k-1} \cdot p$

p = probab. of hitting the center (probab. of a success)

$p = 0.17$

$P(X=8) = 0.83^7 \cdot 0.17 = 0.046$

$$X: \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ & & \dots & & (1-p)^{k-1} \cdot p & \dots \end{pmatrix}$$

Values of X : $\mathbb{N}^*$

3. An oil company conducts a geological study that indicates that an exploratory oil well should have a 20% chance of striking oil. What is the probability that the first strike comes on the third well drilled? What is the probability that the third strike comes on the seventh well drilled?

X - nr. of wells drilled needed to have the first strike

$X \sim Geom(p) = NB(\underline{\underline{1, p}})$

$p = 20\% = 0.20$

$P(X=3) = (1-p)^2 \cdot p = 0.8^2 \cdot 0.2 = 0.128$

Y - nr. of wells drilled needed to have 3 strikes

$Y \sim NB(k, p)$, $P(X=t) = \binom{k-1}{t-1} \cdot p^k (1-p)^{t-k}$

$p = 0.20$, $k = 3$

F S F F S F S

$\underbrace{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6}_{2S}$ 7

$P(Y=7) = \binom{6}{2} \cdot p^3 \cdot (1-p)^4 = 15 \cdot 0.2^3 \cdot 0.8^4 = 0.049$

Exercise 66. Assume that, during each second, a police station receives one call with probability .01 and no calls with probability .99. Use the Poisson approximation to estimate the probability that the operator will miss at most one call if she takes a 5-minute coffee break.

In average, the nr. of calls in a 5-minute interval: $0.01 \cdot 300s = 3$ calls

$X \sim Po(\lambda)$, $P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, $k \geq 0$

X - nr. of calls in a 5-minute interval

$\lambda = 3$ calls/5-mins

$P(X \leq 1) = P(X=0) + P(X=1) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} + e^{-\lambda} \cdot \frac{\lambda^1}{1!} = e^{-\lambda} + \lambda e^{-\lambda} = e^{-3} + 3e^{-3} = 4e^{-3} \approx 0.199$

Ch. 1 (59) 15 chocolate bars
10 children

$\begin{matrix} 3 & 2 & | & | & 4 & 1 & 5 \\ \bullet & \bullet & | & | & \bullet & \bullet & \bullet \\ 1 & 2 & & & 3 & 4 & 5 \\ 1 & 2 & & & 3 & 5 & 4 \end{matrix} \Big\} \; 5!$

(a) the chocolate bars are fungible (interchangeable)

- combinations with replacement : $\binom{n+k-1}{k} = \binom{24}{15}$

$n = 10, \; k = 15$

$\bullet\bullet \,|\, | \,\bullet\bullet\bullet\, | \,\bullet\, | \ldots | \,\bullet\, | \,\bullet$ $\rightarrow$ 9 bars, 15 balls

$9 + 15 = 24$ objects

$\underset{1}{\underline{|}} \;\; \underset{2}{\underline{\bullet}} \;\; \underset{3}{\underline{\bullet}} \;\; \ldots \;\; \underset{24}{\underline{|}}$

(b) $\underset{1}{\underline{\bullet}} \;\; \underset{2}{\underline{\bullet}} \;\; \underset{3}{\underline{\bullet}} \;\; \ldots \;\; \underset{10}{\underline{\bullet}}$ - children

$\Rightarrow$ $15 - 10 = 5$ chocolate bars <u>left</u> $\rightarrow$ 10 children

- combinations with replacement : $n = 10, \; k = 5 \rightarrow \binom{14}{5}$

c) - the chocolate bars are not fungible : $\binom{24}{15} \cdot 15!$

d) - the chocolate bars are not fungible
- each child needs to receive at least one chocolate bar

$X = \{1, 2, 3, \dots, 15\} \longrightarrow \{1, 2, \dots, 10\} = Y \rightsquigarrow$ nr. of surjective functions $f : X \to Y$

$10 \cdot 10 \cdot 10 \quad\quad 10$

10^{15} - nr. of function $f : X \to Y$

10^{15} - nr. of ways to randomly give the chocolate bars to the children

$N = 10^{15} - |\text{at least one child has no bar}| =$ nr. of possibilities to give the chocolate bars to the children such that each child has at least one bar

$|A_1 \cup A_2 \cup \dots A_{10}|$

inclusion-exclusion

A_i : child i has no chocolate bar

$$|A_1 \cup A_2 \cup \dots \cup A_{10}| = \sum_{i=1}^{10} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \dots + (-1)^{11} |A_1 \cap A_2 \cap \dots \cap A_{10}|$$

$$= \sum_{i=1}^{10} 9^{15} - \sum_{i<j} 8^{15} + \sum 7^{15} - \dots$$

$$= 10 \cdot 9^{15} - \binom{10}{2} \cdot 8^{15} + \binom{10}{3} \cdot 7^{15} + \dots$$

$$= \sum_{i=1}^{10} \binom{10}{i} \cdot (10-i)^{15} \cdot (-1)^{i+1}$$

$$N = 10^{15} - \sum_{i=1}^{10} (-1)^{i+1} \binom{10}{i} (10-i)^{15}$$

sgr. 6

Exercise 55. A multiple choice quiz consists of ten questions each with five possible answers from which only one is correct. A students passes the quiz if seven or more of his answers are correct. What is the probability that a student who guesses blindly at all of the questions will pass the test? What is the probability that the student passes the test if at every question he can eliminate three incorrect answers and then guesses between the remaining two?

X - nr. of correct answers

$X \sim Bin(n, p) \Rightarrow P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = \overline{0,n}$

$n = 10$

$p = \frac{1}{5} = 0.2$

$X : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & n \\ & & \dots & & \binom{n}{k} p^k (1-p)^{n-k} & \dots & \end{pmatrix}$

$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$= \binom{10}{7} \cdot 0.2^7 \cdot 0.8^3 + \binom{10}{8} \cdot 0.2^8 \cdot 0.8^2 + \binom{10}{9} 0.2^9 \cdot 0.8^1 + \binom{10}{10} \cdot 0.2^{10} \cdot 0.8^0 = 0.0009$

$p' = \frac{1}{2} = 0.5$

$P(X \geq 7) = \binom{10}{7} \cdot \underbrace{0.5^7 \cdot 0.5^3}_{0.5^{10}} + \dots + \binom{10}{10} \cdot 0.5^{10} = \left[\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right] \cdot 0.5^{10} = 0.172$

- Binomial distribution :
  - count nr. of successes in $\underline{n}$ Bernoulli trials
    (count nr. of Heads in $\underline{n}$ tosses of a coin)
- Geometric distribution :
  - count nr. of trials needed to get the first success
    (nr. of tosses needed to get the first Heads)
- Negative binomial distribution :
  - count nr. of trials needed to get the first $\underline{k}$ successes
    (nr. of tosses needed to get 5 Heads)
- Poisson distribution :
  - nr. of rare events in a time unit
    (nr. of telephone calls in an hour at a call center)

2. 10% of applicants for a job possess the right skills. A company interviews applicants one at a time until they find a qualified applicant.

a) What is the probability that they will interview exactly ten applicants?

b) What is the probability that they will interview at least ten applicants?

a) X - nr. of applicants interviewed by the company

$X \sim Geom(p)$

$$X: \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ & & \dots & & (1-p)^{k-1} \cdot p & \dots \end{pmatrix}$$

$P(X=k) = (1-p)^{k-1} \cdot p$

$$\underbrace{\underset{1}{0}\ \underset{2}{0}\ \underset{3}{0}\ \dots\ \underset{k-1}{0}}_{(1-p)^{k-1}}\ \underset{k}{1} \cdot p$$

$p = \frac{1}{10} = 0.1$

$$P(X=10) = (1-p)^9 \cdot p = 0.9^9 \cdot 0.1 = 0.039$$

b) $$P(X \geq 10) = 1 - P(X < 10) = 1 - [P(X=1) + P(X=2) + \dots + P(X=9)] =$$

$$= 1 - [p + (1-p) \cdot p + (1-p)^2 \cdot p + \dots + (1-p)^8 \cdot p] = 1 - p[\underbrace{1 + (1-p) + (1-p)^2 + \dots + (1-p)^8}_{\text{geometric progression}}] =$$

$$= 1 - p \cdot \frac{1-(1-p)^9}{1-(1-p)} = 1 - \cancel{p} \cdot \frac{1-(1-p)^9}{\cancel{p}} = 1 - 1 + (1-p)^9 = (1-p)^9 = 0.9^9 = 0.387$$

$$1 + r + r^2 + \dots + r^n = 1 \cdot \frac{1-r^{n+1}}{1-r}$$

3. Bob is a high school basketball player. He is a 70% free throw shooter. That means his probability of making a free throw is 0.70. During the season, what is the probability that Bob makes his third free throw on his fifth shot?

What is the probability that Bob needs at least six shots to make three free throws?

What is the probability that Bob makes his first free throw on his fifth shot?

X – nr. of free shots needed to have 3 throws

$X \sim NB(k, p)$

$k=3, \; p=0.7$

$$X: \begin{pmatrix} k & k+1 & \dots & t & \dots \\ & \dots & & \binom{t-1}{k-1} p^k (1-p)^{t-k} & \dots \end{pmatrix}$$

$$P(X=t) = \binom{t-1}{k-1} \cdot p^k \cdot (1-p)^{t-k}$$

$$\underbrace{\underset{1}{1} \quad \underset{2}{0} \quad \underset{3}{1} \quad \dots \quad \underset{t-1}{1}}_{k-1} \quad \underset{t}{1}$$

$$P(X=5) = \binom{4}{2} \cdot p^3 \cdot (1-p)^2 = \binom{4}{2} \cdot 0.7^3 \cdot 0.3^2 = 0.185$$

$$P(X \geq 6) = 1 - P(X<6) = 1 - [P(X=3) + P(X=4) + P(X=5)] =$$

$$= 1 - \left[\binom{2}{2} \cdot p^3 \cdot (1-p)^0 + \underbrace{\binom{3}{2}}_{=3} \cdot p^3 \cdot (1-p) + \binom{4}{2} \cdot p^3 (1-p)^2\right] = 1 - [p^3 + 3 \cdot p^3 (1-p) + 0.185] =$$

$$= 0.502$$

Y - nr. of shots needed to make the first free throw

$Y \sim Geom(p)$

$p = 0.7$

$P(Y=5) = (1-p)^4 \cdot p = 0.3^4 \cdot 0.7 = 0.006$

$Geom(p) = NB(1, p)$

$k = 1$

4. A manufacturer produces VLSI chips, 1% of which are defective. Find the probability that in a box containing 100 chips, no defectives are found, using the approximation given by the Poisson distribution.

- X – nr. of defective chips in the box

$X \sim Bin(n, p)$

$n = 100$

$p = 0.01$

$P(X=0) = \binom{100}{0} p^0 (1-p)^{100} = (1-p)^{100} = 0.99^{100} = 0.366$

Approximate the $Bin(n,p)$ with $Po(\lambda)$ if n – large $(n \geq 30)$ and p – small $(p \leq 0.05)$.

$\lambda = n \cdot p = 100 \cdot 0.01 = 1$

$P(X=0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-\lambda} = e^{-1} \doteq 0.368$

$P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

5. Suppose the average number of lions seen on a 1-day safari is 5. What is the probability that tourists will see fewer than four lions on the next 1-day safari?

X - nr. of lions seen on a 1-day safari

$\lambda = E[X]$ - <u>average</u> nr. of lions seen on a 1-day safari

$\lambda = 5$

$X \sim Po(\lambda)$, $X : \begin{pmatrix} 0 & 1 & 2 & \ldots & k & \ldots \\ & & \ldots & & e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \ldots \end{pmatrix}$ $\qquad P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$$P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) =$$

$$= e^{-\lambda} \cdot \frac{\lambda^0}{0!} + e^{-\lambda} \cdot \frac{\lambda}{1!} + e^{-\lambda} \cdot \frac{\lambda^2}{2!} + e^{-\lambda} \cdot \frac{\lambda^3}{3!} =$$

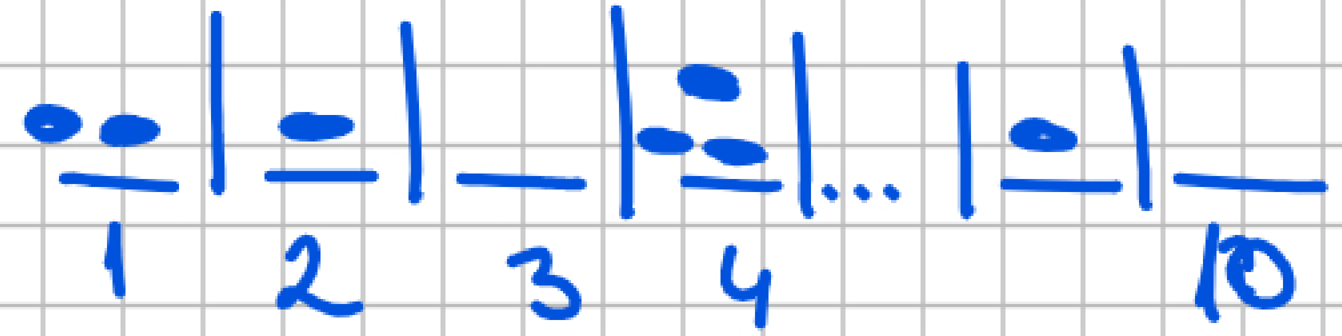
$$= e^{-\lambda} \cdot \left[1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6}\right] = e^{-5}\left[1 + 5 + \frac{25}{2} + \frac{125}{6}\right] = 0.265$$

Ch. 1

59. 15 chocolate bars, 10 children

a) Bose-Einstein pls. : $\binom{n+k-1}{k} = \binom{24}{15} = \binom{24}{9}$

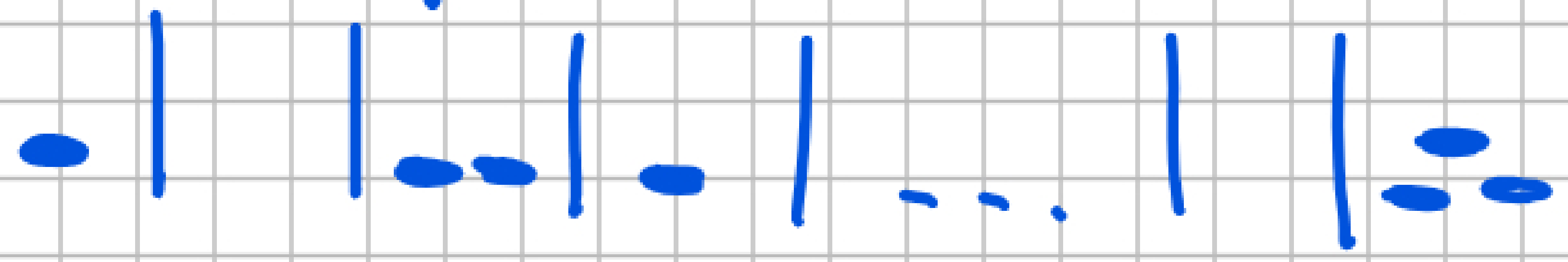
$n=10,\ k=15$



$\Rightarrow$ 9 bars, 15 balls } 24 objects to be arranged on a line

- combinations with replacement

$$\frac{24!}{9!\cdot 15!} = \binom{24}{9}$$



$\rightarrow$ 9 bars, 15 balls

$$\binom{24}{9} = \binom{24}{15}$$

b) $\underset{1}{\bullet} \; \underset{2}{\bullet} \; \underset{3}{\bullet} \; \dots \; \underset{10}{\bullet}$ $\Rightarrow$ 5 chocolate bars to distribute

9 bars, 5 balls $\Rightarrow \binom{14}{9} = \binom{14}{5}$

c) chocolate bars are not fungible (they are distinguishable)

$c_1, c_2, \dots, c_{15}$

10^{15}

$\underline{1\,5} \mid \underline{7} \mid \underline{\quad} \mid \dots \mid \underline{12\,3\,15} \mid \underline{\quad}$

$\underline{7\,1} \mid \underline{5} \mid \dots \mid \underline{\quad} \mid \underline{4\,2\,6} \mid \underline{\quad}$

$\binom{24}{9} \cdot \boxed{15!} = \frac{24!}{9! \cdot 15!} \cdot 15! = \frac{24!}{9!}$

- 1,2 | 3 | _ | .. | 13,14,15 | _

  2,1 | 3 | _ | _ .. | 13,14,15 | _

  3,1 | 2 | _ | _ | 12,4,15 |

d) $10^{15} - |\text{at least one child has no chocolate bar}|$

$$= 10^{15} - \sum_{i=1}^{10} \binom{10}{i} \cdot (10-i)^{15}$$

$$\frac{10}{1}\ \frac{10}{2}\ \frac{}{3}\ \cdots\ \frac{10}{15} \rightarrow 10^{15}$$

- $\binom{10}{1} \cdot 9^{15}$, $\binom{10}{2} \cdot 8^{15}$, ...

- nr. of surjective functions $f : \{1, 2, \ldots, 15\} \rightarrow \{1, 2, \ldots, 10\}$

  $f : A \rightarrow B$, $|B|^{|A|}$

1. Suppose that an airplane engine will fail, when in flight, with probability 1−p independently from engine to engine. Also, suppose that the airplane makes a successful flight if at least 50 percent of its engines remain operative. For what values of p is a four-engine plane preferable to a two-engine plane?

Probability that a 4-engine plane will make a successful flight?
Probability that a 2-engine plane makes a successful flight?

• X – nr. of operative engines during the flight for a 4-engine plan

$X \sim Bin(n, p)$

$n = 4$, p = probab. that an engine will remain operative

$= 1 - P(X < 2) = 1 - [P(X=0) + P(X=1)] = 1 - [\binom{4}{0} p^0 (1-p)^4 + \binom{4}{1} p (1-p)^3]$

$P(X \geq 2) = P(X=2) + P(X=3) + P(X=4) = \binom{4}{2} p^2 (1-p)^2 + \binom{4}{3} p^3 (1-p) + \binom{4}{4} p^4 (1-p)^0 =$

$X : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots & n \\ & & & & \binom{n}{k} p^k (1-p)^{n-k} & & \end{pmatrix}$

$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\underbrace{\underset{1}{1} \quad \underset{2}{1} \quad \underset{3}{0} \quad \dots \quad \underset{n}{0}}_{k \text{ successes}}$

$$= \frac{4!}{2!\cdot 2!}\cdot p^2(1-p)^2 + \frac{4!}{3!\cdot 1!}\cdot p^3(1-p) + \frac{4!}{4!\cdot 0!}\cdot p^4 =$$

$$= \frac{24}{4} p^2(1-p)^2 + 4p^3(1-p) + p^4 = 6p^2(1-2p+p^2) + 4p^3 - 4p^4 + p^4 =$$

$$= 6p^2 - 12p^3 + 6p^4 + 4p^3 - 4p^4 + p^4 = 3p^4 - 8p^3 + 6p^2$$

- Y – nr. of operative engine during a flight for a 2-engine plan

$Y \sim Bin(m', p)$, $m' = 2$ $\quad Y: \begin{pmatrix} 0 & 1 & 2 \\ & - & \end{pmatrix}$ $\quad P(Y=k) = \binom{m'}{k} p^k (1-p)^{m'-k}$

$$P(Y \geq 1) = P(Y=1) + P(Y=2) = \binom{2}{1}\cdot p^1(1-p) + \binom{2}{2} p^2 (1-p)^0 =$$

$$= 2p(1-p) + p^2 = 2p - 2p^2 + p^2 = 2p - p^2$$

$$P(X \geq 2) \overset{>}{\geq} P(Y \geq 1) \iff 3p^4 - 8p^3 + 6p^2 \geq 2p - p^2 \iff 3p^4 - 8p^3 + 7p^2 - 2p \geq 0 \iff$$

$$p(3p^3 - 8p^2 + 7p - 2) \geq 0 \iff p(p-1)^2(3p-2) \geq 0$$

$$\underbrace{p(p-1)^2}_{\geq 0}(3p-2) \geq 0 \iff 3p-2 \geq 0 \iff p \overset{>}{\geq} \frac{2}{3}$$

$p \in [0,1]$

$(p-1)^2 \geq 0$

If $p \in (\frac{2}{3}, 1]$, then the 4-engine plane is preferable

2. A test of weld strength involves loading welded joints until a fracture occurs. For a certain type of weld, 80% of the fractures occur in the weld itself, while the other 20% occur in the beam. A number of welds are tested and the tests are independent.

a) Find the probability that the first beam fracture happens on the third trial or later.

b) Find the average number of trials needed to find the first beam fracture and its variance.

c) Find the probability that the 3rd beam fracture (success) occurs on the 6th trial.

a) X – nr. of trials (tested welds) needed to find the first beam fracture.

$X \sim Geom(p)$

$p = 0.2$

$$X : \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ & \dots & & & (1-p)^{k-1} \cdot p & \dots \end{pmatrix}$$

$P(X=k) = (1-p)^{k-1} \cdot p$

$$\underbrace{\underset{1}{\underline{0}} \quad \underset{2}{\underline{0}} \quad \dots \quad \underset{k-1}{\underline{0}}}_{k-1 \text{ failures}} \quad \underset{k}{\underline{1}}$$

$$P(X \geq 3) = 1 - P(X < 3) = 1 - [P(X=1) + P(X=2)] =$$

$$= 1 - [p + (1-p) \cdot p] = 1 - [0.2 + 0.8 \cdot 0.2] = 0.64$$

b) $$E[X] = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p = p \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = \frac{1}{p} = \frac{1}{0.2} = \frac{10}{2} = 5$$

c) Y – nr. of trials needed to find 3 beam fractures

$Y \sim NB(k, p)$ $k = 3$, $p = 0.2$

$$P(Y=6) = \binom{5}{2} \cdot p^3 (1-p)^3 = \binom{5}{2} 0.2^3 \cdot 0.8^3 = 0.041$$

$$Y : \begin{pmatrix} k & k+1 & k+2 & \dots & s & \dots \\ & & \dots & & \binom{s-k}{k-1} p^k (1-p)^{s-k} & \dots \end{pmatrix}$$

$$\underbrace{\underset{1}{0} \quad \underset{2}{1} \quad \underset{3}{1} \quad \dots \quad \underset{s-1}{0}}_{k-1 \text{ successes}} \quad \underset{s}{1}$$

$$P(Y=s) = \binom{s-1}{k-1} \cdot p^k (1-p)^{s-k}$$

$$s=6, \; k=3, \; p=0.2$$

3. The average number of homes sold by the Acme Realty company is 2 homes per day. What is the probability that exactly 3 homes will be sold tomorrow? What is the probability that more than 4 homes will be sold tomorrow?

X – nr. of homes sold in a day

$X \sim Po(\lambda)$

λ = average nr. of homes sold in a day ($\lambda = E[X]$)

$\lambda = 2$

$$X : \begin{pmatrix} 0 & 1 & 2 & \dots & k & \dots \\ & & \dots & & e^{-\lambda} \cdot \frac{\lambda^k}{k!} & \dots \end{pmatrix} \qquad P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$P(X=3) = e^{-\lambda} \cdot \frac{\lambda^3}{3!} = e^{-2} \cdot \frac{2^3}{3!} = e^{-2} \cdot \frac{\cancel{8}^{4}}{\cancel{6}_{3}} = \frac{4}{3} e^{-2} = 0.180$$

$$P(X>4) = P(X=5) + P(X=6) + \dots = 1 - P(X \le 4) = 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)] =$$

$$= 1 - \left[e^{-\lambda} \cdot \frac{\lambda^0}{0!} + e^{-\lambda} \cdot \frac{\lambda}{1!} + e^{-\lambda} \cdot \frac{\lambda^2}{2!} + e^{-\lambda} \cdot \frac{\lambda^3}{3!} + e^{-\lambda} \cdot \frac{\lambda^4}{4!}\right] = 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \frac{\lambda^4}{24}\right) = 0.053$$

Ch. 2

5

$p_1 + p_2 = \frac{3}{50} + \frac{3}{49}$

$p_1 = \frac{3}{50}, \quad p_2 = \frac{3}{49}$

$P(\text{"third card is an ace"}) = \frac{3}{52-2} = \frac{3}{50}$ (by symmetry)

A: the first card is the Ace of Spades

B: the second card is the 8 of Clubs

C: the third card is an ace

$$P(C \mid A, B) = \frac{P(C \cap A \cap B)}{P(A,B)} = \frac{P(C \cap A \cap B)}{P(A \cap B)} = \frac{\frac{3}{\cancel{52} \cdot \cancel{51} \cdot 50}}{\frac{1}{\cancel{52} \cdot \cancel{51}}} = \frac{3}{50}$$

$$P(A \cap B) = \frac{1}{52} \cdot \frac{1}{51} = \frac{1}{52 \cdot 51} \quad (= P(A) \cdot P(B \mid A))$$

$$P(C \cap A \cap B) = \frac{1}{52} \cdot \frac{1}{51} \cdot \frac{3}{50} = \frac{3}{52 \cdot 51 \cdot 50} \quad (= P(A) \cdot P(B \mid A) \cdot P(C \mid A, B))$$

(32) A : 4, 4, 4, 4, 0, 0

B : 3, 3, 3, 3, 3, 3

C : 6, 6, 2, 2, 2, 2

D : 5, 5, 5, 1, 1, 1

A : the nr. that appears on die A

B : — " — B

⋮

$A > B \equiv A = 4$

(a) $P(A>B) = P(\underline{A=4}) = \frac{4}{6} = \frac{2}{3}$

$P(B>C) = P(\underline{C=2}) = \frac{4}{6} = \frac{2}{3}$

$P(C>D) = P(C=6) + P(C=2, D=1) = \frac{2}{6} + P(C=2)\cdot P(D=1) = \frac{1}{3} + \frac{\cancel{2}}{3}\cdot\frac{\cancel{3}}{\cancel{6}} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

$P(D>A) = P(D=5) + P(D=1, A=0) = \frac{3}{6} + P(D=1)\cdot P(A=0) = \frac{1}{2} + \frac{\cancel{3}}{6}\cdot\frac{\cancel{2}}{\cancel{6}} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$P(A \cap B) = P(A)\cdot P(B) \iff A, B$ – independent events

(b) $A>B$ ind. $B>C$ | "$A=4$", "$C=2$" – independent events

$B>C$ dep. $C>D$ | $C=2$, $C=6 \vee (C=2 \wedge D=1)$ – not independent sgr. 4

3. A pediatrician wishes to recruit 5 couples, each of whom is expecting their first child, to participate in a new natural childbirth regimen. Let p be the probability that a randomly selected couple agrees to participate. If $p = 0.15$, what is the probability that 15 couples must be asked before 5 are found who agree to participate?